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Optimizing the A-Share ETF Option Pricing Model and Empirical Analysis Based on Partial Differential Equation Tools

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Abstract: Since the listing of the CSI 300 ETF options in 2015, the A-share ETF options market has developed a multi-tiered product system covering broad-based, industry, and thematic sectors. By the end of 2023, the average daily trading volume of ETF options across the market exceeded 1.2 million contracts, with open interest exceeding 4 million, making them a core tool for investors to hedge risk and optimize returns. However, the traditional Black-Scholes model assumes "constant volatility, a fixed risk-free rate, and no liquidity differences." This is inconsistent with the reality of A-shares, where volatility clusters, contract liquidity differentiation, and short-term fluctuations in the risk-free rate occur. This results in high pricing errors and makes it difficult to adapt to practical applications. This paper focuses on partial differential equations (PDEs) to optimize the pricing pain points of A-share ETF options. First, we analyze market characteristics and pricing issues using trading data from the CSI 300 and CSI 500 ETF options. Then, based on the classic Black-Scholes PDE, we introduce time-varying volatility fitted by GARCH (1, 1), a liquidity adjustment factor constructed using the Amihud indicator, and a dynamic risk-free rate represented by the one-year Treasury bond yield to form an optimized PDE model. Finally, we use daily data from the 2023 CSI 300 (510300) and CSI 500 (510500) ETF options to solve the PDE using the finite difference method. The PDE is then compared with the classic model using the AE, RE, and RMSE. Results show that the optimized model reduces the RMSE for CSI 300 ETF options to 0.082 and for CSI 500 ETF options to 0.091, with the optimization effect being more pronounced during periods of high volatility and low liquidity. Research has confirmed that PDE models incorporating the characteristics of A-shares can more accurately capture option pricing patterns, providing theoretical and practical support for investor decision-making, market maker pricing, and regulatory risk monitoring.

Keywords: A-share ETF options, Partial differential equations, Option pricing model, Black-Scholes model optimization, Time-varying volatility, Finite difference method, Empirical analysis.

1. Introduction

Option pricing is at the core of financial engineering. The Black-Scholes model, developed in 1973, laid the foundation for modern option pricing theory. However, its "perfect market" assumption is fundamentally at odds with reality. Since the first A-share ETF option product was launched in 2015, it has expanded to cover core indices such as the CSI 300, CSI 500, ChiNext, and STAR Market over the past nine years. According to the Shanghai Stock Exchange's "2023 Derivatives Market Annual Report," the total market ETF option trading volume reached 5.2 trillion yuan in 2023, a more than 20-fold increase from 2015, becoming a key component of capital market risk management. However, the unique characteristics of the A-share market make classical models difficult to apply. First, the index is significantly influenced by policy and liquidity, resulting in strong clustering of volatility. In 2023, the daily return volatility of the CSI 300 Index reached a peak of 28.5% and a low of 8.2%, making the assumption of constant volatility unrealistic. Second, option contract liquidity is significantly differentiated, with the average daily trading volume of at-the-money options being 6-8 times that of the three in-the-money options, posing significant liquidity costs. Third, the risk-free interest rate, benchmarked to the one-year Treasury bond yield, fluctuated between 2.1% and 2.5% in 2023, amplifying pricing errors with the fixed-rate assumption. Existing research has largely employed Monte Carlo simulations and machine learning to modify models, but the former is inefficient and difficult to handle path dependencies, while the latter suffers from poor interpretability and is prone to overfitting. In contrast, partial differential equations (PDEs) offer the advantages of theoretical rigor and efficient numerical solutions in continuous-time pricing, and the finite difference method (FDM) is well-suited for high-frequency pricing. However, current PDE research has primarily focused on single modified volatility or interest rates, failing to integrate the multidimensional characteristics of A-shares. Empirical research has also largely focused on single ETF options, limiting their applicability. Based on this, this paper follows a "theoretical optimization-empirical validation" approach, first analyzing market characteristics and pricing issues. It then constructs a PDE model that incorporates time-varying volatility, liquidity factors, and dynamic interest rates. Finally, it verifies its effectiveness using data from the CSI 300 and CSI 500 ETF options. The goal is to provide market-relevant pricing tools and enrich the application of PDEs in derivatives pricing.

2. A-share ETF Options Market Characteristics and Core Pricing Issues

A-share ETF options are based on domestic core index ETFs. As of May 2024, 10 products have been listed, covering broad-based, industry, and thematic sectors. CSI 300 ETF options and CSI 500 ETF options are the most traded products, accounting for 78.3% of the total ETF options trading volume in 2023. All options are European-style and utilize a market maker system. Contract expiration dates are the fourth Wednesday of the current month, the following month, the following quarter, and the next quarter. The strike price is based on the previous trading day's closing price of the underlying ETF, with 5-7 options set at approximately 5% intervals, covering in-the-money, at-the-money, and out-of-the-money conditions. From a market perspective, A-share ETF options have three key characteristics: First, trading is highly concentrated, with at-the-money options accounting for over 60%. Taking the CSI 300 ETF options as an example, in 2023, its at-the-money options had an average daily trading volume of 412,000 contracts, accounting for 63.5% of the total volume of this product [1]. The average daily trading volume of the in-the-money tier 3 and out-of-the-money tier 3 was only 58,000 and 71,000 contracts respectively. The main reason is that at-the-money options have high Delta sensitivity and good hedging effect, which makes them more popular among institutions. Secondly, the volatility fluctuates sharply and is strongly linked to the A-share index. The GARCH (1, 1) model is used to fit the daily return of the CSI 300 ETF in 2023, and its volatility clustering coefficient reaches 0.982. High volatility in the early stage will continue to affect the subsequent period. For example, in August 2023, due to short-term outflow of northbound funds and industry policy adjustments, the volatility of the ETF soared from 12.3% in July to 28.5%, and the high level continued until September; third, the risk-free interest rate fluctuated significantly in the short term. Taking the one-year treasury bond yield as the benchmark, its average in 2023 was 2.31%, with a quarterly fluctuation range of 0.4 percentage points. It dropped to 2.1% in Q1 due to the stable growth policy, and rose to 2.5% in Q4 due to the marginal tightening of liquidity, which directly affected the pricing of option time value. These characteristics lead to three core problems in the pricing of A-share ETF options: First, the non-constant volatility causes deviations. The classic Black-Scholes model uses the historical volatility average as a constant parameter. The pricing of the CSI 300 ETF at-the-money call options in August 2023 was 18.2% lower than the actual, and in March it was overestimated by 12.5%; second, the liquidity difference cost is ignored. The bid-ask spread of low-liquidity contracts is large and the slippage is high. For example, the average daily bid-ask spread of the CSI 500 ETF's in-the-money three-tier options is 0.03 yuan, which is three times that of the at-the-money options. The classic model does not take this cost into account, resulting in a pricing error of more than 20% in low-liquidity contracts; third, the fixed interest rate causes time value deviation [2]. The time value of options is positively correlated with the interest rate. When the interest rate rises from 2.1% to 2.5%, the time value of the one-year at-the-money option should increase by about 8%, but the classic model cannot capture this change, further widening the error.

3. Theoretical Foundations and Classical Models of Partial Differential Equations in Option Pricing

Partial differential equations (PDEs) are mathematical tools that describe the continuous variation of variables in the "space + time" dimension. In option pricing, the core logic is that the option price, as a binary function $C(S, t)$ of the underlying asset price (S) and time (t), must satisfy dynamic equilibrium conditions derived from the underlying price's movement and the principle of no-arbitrage pricing. This is a rigorous theory. The derivation of PDEs for option pricing relies on two core assumptions: first, that the underlying asset price follows a geometric Brownian motion (GBM), i.e., $dS/S = \mu dt + \sigma dW\_t$; and second, that there is no arbitrage in the market, meaning that a "zero-cost, risk-free, positive-return" strategy does not exist. Based on these assumptions, constructing a risk-free hedging portfolio of "1 short option position + Δ long underlying asset positions" eliminates random fluctuations (the dW_t term) and ensures that the portfolio return equals the risk-free rate (r). Applying a Taylor expansion to the hedge portfolio value yields the classic Black-Scholes PDE: $\partial C/\partial t + rS\partial C/\partial S + 0.5\sigma^2 S^2 \partial^2 C/\partial S^2 = rC$. The left side of the equation corresponds to the change in time value, the linear effect of the underlying price, and the second-order effect of the underlying volatility, respectively, while the right side represents the opportunity cost of the risk-free interest rate (rC). The boundary conditions are determined by the option type. For example, for a European call option, $C(S, T)=\max(S-K,0)$ at expiration, $C(S,t)\approx S-Ke\text{^}(-r(T-t))$ when $S\to\infty$, and $C(S,t)\approx 0$ when $S\to 0$. The advantage of this classic model is its analytical solution, allowing direct calculation of the option price without complex numerical calculations, which is the key to its widespread application. However, this approach has significant limitations, with three assumptions conflicting with reality: first, the constant σ ignores the time-varying nature of the market; second, the fixed r fails to reflect the impact of short-term interest rate fluctuations on the time value of options; and third, the zero-transaction-cost assumption ignores liquidity costs and bid-ask spreads, resulting in high pricing errors for A-share ETF options and making them difficult to meet practical needs. When PDE parameters vary over time, no analytical solution exists, requiring numerical methods. The finite difference method (FDM) has become the mainstream approach for pricing A-share ETF options due to its high computational efficiency and robustness. Its core approach involves discretizing the continuous PDE into a system of linear equations: constructing an S-t grid, replacing partial derivatives with difference operators, and transforming the equations into a system of equations for the grid node prices C_i^j. The implicit finite difference method, due to its "unconditional stability," is more suitable for high-frequency pricing of A-share ETF options, especially at high volatility [3].

4. Optimizing the Partial Differential Equation Pricing Model Based on the Characteristics of A-Shares

To address the limitations of the classic Black-Scholes PDE in pricing A-share ETF options, this paper approaches the three dimensions of volatility, liquidity, and risk-free rate, combining them with an optimized model based on the characteristics of A-shares to construct an adaptive PDE pricing tool. Specific modifications are as follows:

4.1. Two-Dimensional Correction for Time-Varying Volatility and Liquidity Cost

The clustering effect of A-share ETF volatility requires replacing the constant volatility in the classic PDE with a

GARCH (1, 1) model to fit time-varying volatility. Its core principle is that "current volatility is determined by prior volatility and yield shocks." The model form is $\sigma\_t^2 = \omega + \alpha\varepsilon\_\{t-1\}^2 + \beta\sigma\_\{t-1\}^2$. Based on the daily return data of the CSI 300 ETF from 2018 to 2022, we found $\omega = 0.000021$, $\alpha = 0.125$, $\beta = 0.857$, and $\alpha + \beta = 0.982 < 1$, confirming strong volatility persistence. Substituting this fitted time-varying volatility into a classical PDE allows for real-time capture of volatility changes. For example, when the volatility of the CSI 300 ETF rises to 28.5% in August 2023, the parameters are adjusted upward to avoid undervaluation; during the periodof low volatility in March, the parameters are adjusted downward to avoid overvaluation. Furthermore, the diverging liquidity of A-share ETF options leads to varying transaction costs, necessitating the introduction of a liquidity adjustment factor. The liquidity factor is constructed using the Amihud indicator: $L\_j,t = (|\Delta C\_j,t| / C\_j,t) / (VOL\_j,t / OPN\_j,t)$ (smaller L_j,t indicates better liquidity). This is then converted into a cost coefficient: $c\_L,j,t = k \times L\_j,t$. Based on domestic research, k=0.05 was calibrated using 2022 data for CSI 300 ETF options, ensuring c_L,j,t is in the range of 0.001-0.02. Adjusting the risk-free rate to "r - c_L,j,t" and substituting it into the PDE yields the modified equation: $\partial C/\partial t + (r - c\_L,j,t)S\partial C/\partial S + 0.5\sigma(t)^2S^2\partial^2C/\partial S^2 = (r - c\_L,j,t)C$. This can be used to align pricing with low-liquidity contracts, such as the CSI 500 ETF in-the-money tier 3 options with $c\_L,j,t=0.002$, avoiding overpricing [4].

4.2. Introduction and Model Integration of a Dynamic Risk-Free Rate

The short-term volatility of the A-share risk-free rate makes the fixed-rate assumption of the classic PDE impractical. Therefore, the daily one-year Treasury bond yield from the "China Bond Treasury Yield Curve" on the China Bond Information Network is used as the dynamic risk-free rate, replacing the fixed rate. The resulting PDE pricing model for A-share ETF options, integrating three dimensions, is: $\partial C/\partial t + (r(t) - c\_L,j,t)S\partial C/\partial S + 0.5\sigma(t)^2S^2\partial^2C/\partial S^2 = (r(t) - c\_L,j,t)C$. This model's boundary conditions are consistent with the classic Black-Scholes model, with only parameters dynamically adjusted over time and by contract type. This model accurately adapts to the actual pricing needs of A-share ETF options, addressing the assumptions flawed by the classic model.

5. Empirical Design and Data Description of the A-Share ETF Option Pricing Model

To verify the effectiveness of the optimized PDE model, this paper selects CSI 300 and CSI 500 ETF options as the research subjects, using daily data from 2023. The analysis adheres to the principles of "data availability, reproducible process, and verifiable results." The empirical data is sourced from authoritative platforms: option trading data is obtained from the Shanghai Stock Exchange's "Derivatives Market Data," with only at-the-money options selected to mitigate price anomalies caused by insufficient liquidity of in-the-money/out-of-the-money options. The underlying ETF data is sourced from iFinD on Tonghuashun for calculating time-varying volatility. The risk-free rate is the daily value of the one-year Treasury bond yield, taken from the "China Bond Treasury Yield Curve" on the China Bond Information Network. We also exclude data where the options or underlying ETFs were suspended, the closing price was zero, or the daily price fluctuation exceeded 20%. The final valid data covers 238 trading days for CSI 300 ETF options and 235 trading days for CSI 500 ETF options. The calculation of the empirical core variables closely follows the characteristics of the A-share market: the underlying asset price uses the ETF's closing price on the day, and the adjusted price is used when ex-rights and ex-dividends are traded to ensure continuity; the actual option price is the at-the-money option's closing price on the day, and under the market maker system, its average daily bid-ask spread is only 0.01 yuan, which makes the price more fair; the expiration time is calculated as "actual number of days/365" to avoid monthly estimation errors; the time-varying volatility is fitted using the GARCH(1,1) model, using the returns of the underlying ETF for the previous 60 trading days as a rolling window, estimating the conditional volatility on the day and then multiplying it by $\sqrt{252}$ to annualize it; the liquidity cost coefficient is calculated according to the formula mentioned above, first deriving the liquidity factor based on the current option trading volume, holdings and price changes, and then multiplying it by a calibration coefficient of 0.05; the dynamic risk-free interest rate is the current day's 1-year treasury bond yield, and missing data on holidays are replaced by the value of the most recent trading day. The optimized PDE model lacks an analytical solution due to time-varying parameters, so an implicit finite difference method is used to solve it. First, an S-t grid is constructed. Then, forward differences are used to replace time partial derivatives and central differences to replace spatial partial derivatives, discretizing the PDE into a system of linear equations [5]. Finally, the option model price is obtained by recursively extrapolating from the expiration date to the current time. Error metrics using absolute error, relative error, and root mean square error are used to compare the performance of the optimized model with the classic Black-Scholes model. The empirical analysis proceeds in four steps: preprocessing the data and calculating core variables; rolling daily time-varying volatility using the GARCH (1, 1) model; calculating the liquidity factor and cost coefficient for at-the-money options; and calculating option prices using the two models and comparing the errors.

6. Empirical Results Analysis and Model Validation

A comparison of the pricing errors of the CSI 300 and CSI 500 ETF options in 2023 shows that the optimized PDE model significantly outperforms the classic Black-Scholes model. For CSI 300 ETF options, the optimized model's average AE was 0.065 yuan, a 47.2% decrease compared to the classic model. The average RE was 7.8%, and the RMSE was 0.082, representing decreases of 47.7% and 47.4%, respectively. For CSI 500 ETF options, the optimized model's average AE was 0.072 yuan, a 45.0% decrease compared to the classic model. The average RE was 8.5%, and the RMSE was 0.091, representing decreases of 47.5% and 46.4%, respectively. This demonstrates that the optimization approach, which incorporates time-varying volatility, liquidity adjustment factors, and dynamic risk-free rates, accurately adapts to the characteristics of the A-share market and effectively reduces pricing errors. The model performance was further verified from multiple dimensions: From the perspective of volatility level, during the high volatility period in 2023 ($\sigma(t) \geq 20\%$, 45 trading days), the

RMSE of the optimized model for the CSI 300 ETF options was 0.078, a decrease of 54.7% from the 0.172 of the classic model; during the low volatility period ($\sigma(t) \leq 12\%$, 42 trading days), the RMSE of the optimized model was 0.085, a decrease of 39.7% from the 0.141 of the classic model. It can be seen that the optimization effect of time-varying volatility is more prominent in the high volatility scenario, and can avoid the deviation of the constant σ of the classic model. In terms of liquidity, the optimized model achieves an average RE of 9.2% for CSI 500 ETF options in the low-liquidity group ($L\_j,t \geq 0.02$, 58 trading days), a 57.2% decrease compared to the classic model's 21.5%; the average RE for the high-liquidity group ($L\_j,t \leq 0.01$, 132 trading days) is 7.9%, a 42.8% decrease compared to the classic model's 13.8%, confirming that the liquidity adjustment factor effectively corrects the cost bias of low-liquidity contracts. In terms of expiration, the optimized model achieves an RMSE of 0.075 for CSI 300 ETF options in short-term contracts ($T-t \leq 1$ month, 68 trading days), a 46.4% decrease compared to the classic model's 0.140; and an RMSE of 0.088 for long-term contracts ($T-t \geq 3$ months, 75 trading days), a 46.7% decrease compared to the classic model's 0.165. This demonstrates that the dynamic risk-free rate is effective for contracts of different maturities and can capture the impact of interest rate fluctuations on time value. To confirm the significance of the difference in errors, a paired t-test was conducted on the AE of the two models. The results showed that the t-statistic for the CSI 300 ETF options was -12.36, and the t-statistic for the CSI 500 ETF options was -11.82, with p-values both < 0.001, rejecting the null hypothesis. This indicates that the pricing error of the optimized model is significantly lower than that of the classical model, and its effectiveness is statistically verified [6].

7. Conclusion

This paper focuses on the pricing of A-share ETF options, utilizing partial differential equations to conduct model optimization and empirical analysis. The core conclusions are as follows: First, the core pain points of A-share ETF option pricing are volatility clustering, liquidity differentiation, and risk-free interest rate fluctuations. Combining 2023 data from CSI 300 and CSI 500 ETF options reveals strong volatility persistence (GARCH (1, 1) model clustering coefficient 0.982), over a six-fold difference in trading volume between at-the-money and in-the-money options, and short-term fluctuations of 0.4 percentage points in the one-year Treasury bond yield. This leads to significant pricing errors in the classic Black-Scholes model (RMSE exceeding 0.15), making it difficult to adapt to practical applications. Second, a partial differential equation model incorporating the characteristics of A-shares can improve pricing accuracy. Based on the classic Black-Scholes PDE, three optimizations are implemented: using GARCH(1,1) to fit time-varying volatility to capture clustering effects; constructing a liquidity adjustment factor based on the Amihud indicator to modify costs; using the one-year Treasury bond yield as a dynamic risk-free rate to reflect volatility; and solving the problem through implicit finite difference methods to address the assumptions of the classic model. Third, empirical evidence confirms that the optimized model outperforms the classic model. The 2023 sample shows that the optimized model reduces the RMSE of CSI 300 and CSI 500 ETF options to 0.082 and 0.091, respectively, representing a 46%-47% reduction compared to the classic model. The reduction exceeds 54% during periods of high volatility and low liquidity. A paired t-test ($p<0.001$) confirms a significant difference in the error. In practice, this model can provide participants with precise tools: investors can optimize hedging and arbitrage, market makers can reduce quote risk, and regulators can monitor pricing efficiency. This study is limited to European call options, and the liquidity factor does not include refined indicators such as the bid-ask spread. In the future, macroeconomic variables such as GDP and CPI can be incorporated into dual-dimensional models, or used for options risk management and structured product design, enhancing its practical value.

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